ON WELL-POSEDNESS OF THE LINEAR CAUCHY PROBLEM WITH THE DISTRIBUTIONAL RIGHT-HAND SIDE AND DISCONTINUOUS COEFFICIENTS

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Abstract. The classical result on well-posedness of Cauchy problem for the linear ordinary differential system with the distributional right-hand side and smooth matrix of coefficients plays fundamental role in many applications of distribution theory to ordinary and partial differential equations. In the present paper we generalize this result to the case of system

$$x' - A(t)x = f, \tag{0.1}$$

where f is a distribution, $A \in \mathbb{G}^\infty$, $\mathbb{G}^\infty$ is the space of functions possessing at most first-kind discontinuities together with all their derivatives defined almost everywhere. The left-hand side of system (0.1) contains the product of a distribution and, in general, a discontinuous function, which is undefined in the classical space $\mathcal{D}'$ of distributions with the smooth test functions. As a result, the Cauchy problem for (0.1) in general has no solution in $\mathcal{D}'$. In what follows, we consider system (0.1) in the space $\mathcal{R}'$ of distributions with $\mathbb{G}^\infty$-test functions, whose elements admit continuous multiplication by functions in $\mathbb{G}^\infty$, and show that there exists the unique solution of the Cauchy problem for (0.1) which depends continuously on f. The proof of this result requires investigation of structure of the kernel of operator of restriction of distributions from $\mathcal{R}$ to $\mathcal{D}$, of properties of operation of multiplication and of properties of multi-valued (yet, in a sense, continuous) operation of differentiation in $\mathcal{R}'$.

1. Introduction

The theory of distributions, which made possible to tackle in a systematic and mathematically rigorous way the discontinuous solutions of ordinary and partial differential equations, appeared in its finished form in 1950 in the famous monograph by L. Schwartz [Sch50] that contained, among other fundamental results of the new theory, the result on well-posedness of the Cauchy problem for the linear differential system with the smooth matrix of coefficients and distributional right-hand side. It is the generalization of this result to the case of system

$$x' - A(t)x = f, \tag{1.1}$$

where f is a distribution and A may have at most first-kind discontinuities together with all its derivatives defined almost everywhere, which occupies a central place in the present work.

The linear differential equation (1.1) is an example of an ordinary differential equation whose left-hand side may contain the product of a distribution and a discontinuous function. The study of certain classes of such differential equations started shortly after creation of the distribution theory, and was inspired both by efforts to extend the field of applicability of the theory of distributions (e.g., the study of linear ordinary differential equations with the distributional coefficients of the order of singularity $\leqslant 1$ in [Kur59, Kur58], also, see [Fil88, Tvr02, PT93]) as well as by numerous problems of optimal control with the unimodal phase restrictions on control, where the product of a distributional optimal control and the corresponding discontinuous solution of the differential equation arise (see [Mil93, SZ97, Kin07]).

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The major obstacle on the way of construction of systematic theory of such differential equations in the classical framework of the theory of distributions (by which we mean the theory of the space of linear continuous functionals defined on a certain space of the smooth test functions, e.g., the theory of the space $\mathcal{D}'$ [GS64]) – that is, the impossibility of continuous multiplication of distributions by discontinuous functions (see [DK07a, DK06]) – is closely related to the requirement of the smoothness of the test functions (which is crucial for the classical "$(f', \varphi) = -(f, \varphi')$" definition of the distributional derivative [GS64]). For example, in (1.1) the product Ax, determined by the equality (let us consider the scalar case for simplicity)

$$(Ax, \varphi) = (x, A\varphi), \tag{1.2}$$

where φ is a smooth test function, is undefined, since $A\varphi$ is in general no longer smooth. Formally, there is no need in use of the equality (1.2) to define the value of the product Ax: there are various definitions of the product Ax in the space $\mathcal{D}'$ such that Ax coincides with the classical product if A is smooth (see [Sar94, Sar95, Sar03], where the family of distributional products which are invariant under unimodular transformations and satisfy the Leibniz product rule is successfully applied to study of certain classes of ordinary and partial differential equations; see [PT93, Fil88] and further references therein for other definitions of the product of a distribution and a discontinuous function in the space $\mathcal{D}'$). Nevertheless, the inevitable lack of continuity in the topology of $\mathcal{D}'$ of any definition of the product of, in particular, Heaviside function and Dirac delta-function makes every such definition unacceptable for our proof of the well-posedness of the Cauchy problem for (1.1). It is the space of distributions with the discontinuous test functions where (1.2) defines the continuous operation of multiplication of distributions by discontinuous functions which can provide proper meaning for the product Ax and, thus, for the system (1.1).

The space of distributions with the test functions that are infinitely differentiable outside the origin where they possess both one-sided limits (together with all their derivatives defined outside the origin) was introduced in [Kur96] in application to the problems of construction of self-adjoint operators corresponding to finite-rank perturbations of the n-th derivative operator with the support at the origin (also, see [KB98]). The generalization to the case of test functions of several variables possessing, in certain extent, an arbitrary set of points of discontinuity, was obtained in [DK07b] in order to provide existence of Nash equilibrium for a class of zero-sum games with first-kind discontinuous payoff functions. Before we define our space of distributions with discontinuous test functions (in the next section), let us provide certain heuristics on multiplication of Dirac delta-function by Heaviside function. Let $\rho \in \mathbb{L}(\mathbf{R})$ be such that

$$\operatorname{supp}(\rho) \subset (-1, 1) \quad \text{and} \quad \int_{\mathbf{R}} \rho(t)dt = 1.$$

We define delta-family $\{\rho_\tau^\varepsilon\}_{\varepsilon>0}$ by the formula

$$\rho_\tau^\varepsilon(t) = \frac{1}{\varepsilon}\rho\left(\frac{t-\tau}{\varepsilon}\right), \quad t \in \mathbf{R},$$

so that $\rho_\tau^\varepsilon \to \delta_\tau$ in $\mathcal{D}'$ as $\varepsilon \to 0+$. Let θ_τ be Heaviside function discontinuous at τ. Then

$$\theta_\tau \rho_\tau^\varepsilon \to \alpha\delta_\tau$$

in $\mathcal{D}'$, where the complex coefficient $\alpha \in \mathbf{C}$, as straightforward calculations show, is given by

$$\alpha = \int_0^\infty \rho(t)dt. \tag{1.3}$$

Thus, in order to avoid multi-valuedness of the product of delta-function and Heaviside function it is necessary to specify "additional information" on δ_τ, which is impossible in the classical space $\mathcal{D}'$ of distributions with the smooth test functions, but which is, however, possible in the space $\mathcal{R}'$ of distributions defined on the space of discontinuous test functions – the space of

functions which have compact support and which are regulated together with their derivatives of all orders defined almost everywhere (function is called regulated if it possesses both one-sided limits at every point of the interval [Die69]): there is delta-function defined by the formula

$$(\delta_\tau^\alpha, \varphi) = \alpha\varphi(\tau+) + (1-\alpha)\varphi(\tau-),$$

so

$$\theta_\tau \delta_\tau^\alpha = \alpha\delta_\tau^+,$$

where $(\delta_\tau^+, \varphi) = \varphi(\tau+)$, and $\rho_\tau^\varepsilon \to \delta_\tau^\alpha$ if and only if (1.3) holds. In a similar way we may define the derivatives of the delta-function

$$(\delta_\tau^{(k)\alpha}, \varphi) = \alpha\varphi^{(k)}(\tau+) + (1-\alpha)\varphi^{(k)}(\tau-),$$

so that $(\rho_\tau^\varepsilon)^{(k)} \to \delta_\tau^{(k)\alpha}$ if and only if the equality (1.3) holds.

Now, being extending major constructions of the classical distribution space $\mathcal{D}'$ to the space $\mathcal{R}'$ of distributions with the discontinuous test functions, one immediately encounters the impossibility to employ the classical definition of the derivative

$$(f', \varphi) = -(f, \varphi'), \tag{1.4}$$

even when φ' is assumed to be defined almost everywhere: first, in this assumption, (1.4) gives rise to a non-linear operator of differentiation, second, as it follows from the above considerations, each distribution possesses whole family of its derivatives, e.g., fix arbitrary $\alpha \in \mathbf{C}$ and consider delta-family $\{\rho_\tau^\varepsilon\}_{\varepsilon>0}$ which corresponds to a locally absolutely continuous function ρ such that (1.3) holds, to obtain that the delta-function δ_τ^α is the derivative of θ_τ; moreover, if f is a distribution, and f' is its derivative, then $f' + c(\delta_\tau^1 - \delta_\tau^0)$ is another derivative of f for every c and every τ (formally, in [Kur96, KB98] the definition (1.4) with φ' defined outside the origin is used; however, since the derivatives of the distributions which are considered in [Kur96, KB98] are specified in accordance with the natural multi-valuedness of the differentiation operator, the insufficiencies of the definition (1.4) are not crucial for the particular results in [Kur96, KB98]; in [DK07b] the problem of definition of derivative was in fact avoided, since all elements of the distribution space in [DK07b] are, in a sense, measure-type distributions).

It is natural to require from the definition of the derivative of a distribution to possess the following property of continuity: if f is the distribution, and g is its derivative, then there exists a family $\{f_\varepsilon\}_{\varepsilon>0}$ of locally absolutely continuous functions such that

$$f_\varepsilon \to f, \quad f'_\varepsilon \to g$$

as $\varepsilon \to 0+$, and conversely, if $f_\varepsilon \to f$, and there exists a distribution g such that $f'_\varepsilon \to g$, then the distribution g is the derivative of f (in fact, since this is the only requirement which we impose on the definition of the derivative, this property is *already* a definition of the derivative).

Thus, the aims of the present paper are the following.

1) To give the (analytical) definition of the derivative which possesses the aforementioned property of continuity.

2) To prove the well-posedness of the Cauchy problem for the linear system

$$x' - A(t)x = f,$$

where f is a distribution, and A is regulated together with all its derivatives defined almost everywhere (and, thus, in general discontinuous).

We would like to emphasize the fact that the operation of differentiation in $\mathcal{R}'$ is multivalued. The formulations of the statements which imply required well-posedness of Cauchy problem (Theorems 3.1 – 3.5 below) coincide with the formulations of the analogous statements for the space $\mathcal{D}'$ in [Shi84] (see [Sch50]), but the aforementioned multi-valuedness of the operation of differentiation requires new (yet natural) definition of the solution as well as totally new proofs.

In fact, the formulation of Theorems 3.1 – 3.5 allows us to write down the solution of the Cauchy problem explicitly, as the following examples show (the proofs of Examples 1.1 and 1.2 are provided in Section 3).

Example 1.1. Let us consider in the space $\mathcal{R}'$ the following Cauchy problems:

$$x' = a\theta_\tau x + b\delta^\alpha_\tau, \quad x = 0 \text{ if } t < t_0, \tag{1.5}$$

$$x' = a\theta_\tau x + b\delta'^\alpha_\tau, \quad x = 0 \text{ if } t < t_0, \tag{1.6}$$

$$x' = a\theta_\tau x + b\delta''^\alpha_\tau, \quad x = 0 \text{ if } t < t_0, \tag{1.7}$$

where $a,\, b \in \mathbf{C}$, $\tau,\, t_0 \in I$, $t_0 < \tau$. Let us define $\zeta_\tau(t) = t - \tau$ for $t > \tau$, $\zeta_\tau(t) = 0$ for $t < \tau$. We call the solution of the differential equation (1.5), (1.6), (1.7) the distribution $x \in \mathcal{R}'$ which possesses the derivative $x' \in \mathcal{R}'$ such that after the substitution of x' and x in (1.5), respectively, in (1.6), (1.7), the equation becomes the identity $\mathcal{R}'$. Using Theorem 3.2 below, we find that the solution of the Cauchy problem (1.5) is the ordinary function given by the formula

$$x = be^{a\zeta_\tau}\theta_\tau$$

(note that right-hand side of (1.5) does not contain the product of a singular distribution and a discontinuous function, and the solution of the Cauchy problem does not depend on $\alpha \in \mathbf{C}$). Further, the solution of the Cauchy problem (1.6) is the distribution given by the formula

$$x = e^{a\zeta_\tau}(ab\alpha\theta_\tau - b\delta^\alpha_\tau),$$

and the distribution

$$x = e^{a\zeta_\tau}(\alpha a^2 b\theta_\tau - 2\alpha ab\delta^+_\tau + b\delta'^\alpha_\tau)$$

is the solution of the Cauchy problem (1.7). Note that the solutions of the Cauchy problems (1.6) and (1.7) *depend* on the value of $\alpha \in \mathbf{C}$.

Example 1.2. Suppose that $I = \mathbf{R}$, and we are given a countable set $\Upsilon \subset (0, \infty)$. Let $\{a_\gamma\}_{\gamma\in\Upsilon}$, $\{b_\gamma\}_{\gamma\in\Upsilon} \subset \mathbf{C}$ be such that $\sum_{\gamma\in\Upsilon} |a_\gamma|$, $\sum_{\gamma\in\Upsilon} |b_\gamma| < \infty$. We define

$$a = \sum_{\gamma\in\Upsilon} a_\gamma\theta_\gamma, \quad b = \sum_{\gamma\in\Upsilon} b_\gamma\theta_\gamma.$$

Let us consider the following Cauchy problem

$$x' = a(t)x + b'', \quad x = 0 \text{ for } t < 0, \tag{1.8}$$

the second derivative of b is specified by the formula $b'' = \sum_{\gamma\in\Upsilon} b_\gamma \delta'^{\alpha(\gamma)}_\gamma$, where $\alpha : I \mapsto \mathbf{C}$ is a bounded continuous function (Cauchy problem (1.6) is a special case of Cauchy problem (1.8)). The solution of the Cauchy problem (1.8) is the distribution given by the formula

$$x = \sum_{\gamma\in\Upsilon} b_\gamma \exp\left(\int_\gamma^t a(s)ds\right)\Big(\big(\alpha(\gamma)a(\gamma+) + \big(1-\alpha(\gamma)\big)a(\gamma-)\big)\theta_\gamma - \delta^{\alpha(\gamma)}_\gamma\Big).$$

Observe that x does not depend on values of α at the points of Υ where $b_\gamma \neq 0$ if and only if a is continuous on $\Upsilon \cap \{\gamma : b_\gamma \neq 0\}$ (e.g., when Υ is the Cantor set, and a is the Cantor function).

As follows from Theorem 3.5 below, the solutions of the Cauchy problems (1.5)–(1.8) can also be obtained if the delta-functions and their derivatives in the-right hand sides of the differential equations in (1.5)–(1.8) are replaced by converging families of locally-summable functions.

We also consider linear differential equations of higher orders, that is,

$$X^{(m)} - A_{m-1}X^{(m-1)} - \cdots - A_0 X = F, \tag{1.9}$$

where A_i are the matrix-valued functions which are regulated (together with their derivatives of all orders defined almost everywhere) and F is a matrix-valued distribution in $\mathcal{R}'$. We prove the well-posedness of the Cauchy problem for (1.9) (Theorems 3.3'–3.5'), and so, is a sense,

generalize the result in [Der86] where the technique of quasidifferential equations is used to obtain the sufficient conditions on distribution $F \in \mathcal{D}'$ and coefficients A_k for existence and uniqueness of the locally-summable solution of the Cauchy problem for (1.9).

In what follows, we show that every distribution in $\mathcal{D}'$ admits a linear continuous extension from $\mathcal{D}$ to $\mathcal{R}$ (Theorem 2.8). Using Gelfand homomorphism induced by Banach algebra of regulated functions, we show that the space of distributions $\mathcal{R}'$ is isomorphic to the space $\mathcal{D}'(I_*)$ of distributions with the smooth test functions defined on a totally-disconnected Hausdorff space (Theorem 2.19), so in this sense the use of the term "discontinuous test functions" is purely conventional (nevertheless, the space of distributions $\mathcal{D}'(I_*)$ seems to be not studied before).

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2. Distributions

Let us start with the definitions of functions spaces and functions algebras which are used throughout this paper. We denote by $\mathbb{L}_{\text{loc}} = \mathbb{L}_{\text{loc}}(I)$ the linear space of locally-summable functions $I \mapsto \mathbf{C}$, where $I = (a, b) \subset \mathbf{R}$ is an open interval (in particular, $I = \mathbf{R}$). Let $\mathbb{L}_\infty = \mathbb{L}_\infty(I)$ be the algebra of functions essentially bounded on I, endowed with the norm

$$\|g\|_{\mathbb{L}_\infty} = \text{esssup}_{t\in I}|g(t)|.$$

In what follows, we denote by $\mathbb{G} \subset \mathbb{L}_\infty$ the algebra of *regulated functions*, that is, the algebra of functions $g : I \mapsto \mathbf{C}$ possessing both one-sided limits

$$g(t-) := \text{esslim}_{s\to t-} g(s), \quad g(t+) := \text{esslim}_{s\to t+} g(s)$$

for every $t \in I$ (equivalently, possessing at most first-kind discontinuities on I). As is well-known, $\mathbb{G}$ is a Banach algebra [Die69].

Theorem 2.1 ([Der02])**.** *For every $g \in \mathbb{G}$ the set of points of discontinuity $T(g) := \{t \in I : \sigma_t(g) := g(t+) - g(t-) \neq 0\}$ is at most countable.*

We define support of the function $g \in \mathbb{G}$ to be the set

$$\text{supp}(g) = \text{cl}\{t \in I : g(t-) \neq 0 \text{ or } g(t+) \neq 0\},$$

where cl stands for the closure in I.

Further, let us denote by $\mathbb{G}^\infty \subset \mathbb{L}_\infty$ the subalgebra of functions $g \in \mathbb{G}$ such that for every $k \in \mathbf{N}$ there exists a regulated function $g^{(k)} \in \mathbb{G}$ (called the k-th derivative of g) such that

$$g^{(k)}(t\pm) = \operatorname*{esslim}_{s\to t\pm,\ s\neq t} \left(\frac{g^{(k-1)}(s) - g^{(k-1)}(t\pm)}{s - t} \right), \quad t \in I,$$

where $g^{(0)} := g$. As follows from the remark above, for every $g \in \mathbb{G}^\infty$, $k \in \mathbf{N}_0$, the set of points of discontinuity of the k-th derivative $T(g^{(k)}) := \{t \in I : \sigma_t(g^{(k)}) \neq 0\}$ is at most countable. We endow algebra $\mathbb{G}^\infty$ with the countable family of norms

$$\|g\|_k = \max_{0\leqslant i\leqslant k} \|g^{(i)}\|_{\mathbb{L}_\infty}, \quad k \in \mathbf{N}_0 := \mathbf{N} \cup \{0\}.$$

If we denote by $\mathbb{C} \subset \mathbb{L}_\infty$ the algebra of bounded continuous functions $I \mapsto \mathbf{C}$, then $\mathbb{C}$ is a proper subalgebra of $\mathbb{G}$. Furthermore, if $\mathbb{C}^\infty \subset \mathbb{L}_\infty$ is the algebra of bounded infinitely (continuously) differentiable functions, then $\mathbb{C}^\infty$ is a proper subalgebra of $\mathbb{G}^\infty$ (we add "continuously" here since the elements of algebra $\mathbb{G}^\infty$ are also "infinitely differentiable" in the above sense).

In what follows, we denote by $\mathcal{D} = \mathcal{D}(I)$ the classical space of $\mathbb{C}^\infty$-test functions. Let $\mathcal{D}' = \mathcal{D}'(I)$ be the space of linear continuous functionals $\mathcal{D} \mapsto \mathbf{C}$ (called *distributions*), endowed with weak* topology (see [GS64, Shi84]).

We proceed now to the definition of the space of $\mathbb{G}^\infty$-test functions, containing as a subspace the classical space of $\mathbb{C}^\infty$-test functions. Namely, let $\mathfrak{J} = \{J\}$ be the family of all subintervals

in $\mathbf{R}$ such that $\bar{J} \subset I$. Every $\varphi \in \mathbb{G}^\infty(J)$, $J \in \mathfrak{J}$, can be extended to I by assigning zero values on $I \setminus J$, so we may define

$$\mathcal{R} = \bigcup_{J \in \mathfrak{J}} \mathbb{G}^\infty(J)$$

– the linear space consisting of functions $\varphi \in \mathbb{G}^\infty$ possessing compact support in I. We endow $\mathcal{R} = \mathcal{R}(I)$ with the locally-convex topology of inductive limit of countably-normed spaces $\mathbb{G}^\infty(J)$, $J \in \mathfrak{J}$, and call its elements analogously as *test functions*.

The following theorem gives the description of the topology in $\mathcal{R}$ in terms of convergent sequences (its proof, as well as the proof of Lemma 2.8 and the proof of Theorem 2.5, is provided in the last section).

Lemma 2.2. *Given $\{\varphi_k\}_{k=1}^\infty \subset \mathcal{R}$ and $\varphi \in \mathcal{R}$, we have that $\varphi_k \to \varphi$ in $\mathcal{R}$ if and only if $\varphi_k \to \varphi$ in $\mathbb{G}^\infty$ and there exists $J \in \mathfrak{J}$ such that* $\operatorname{supp}(\varphi_k) \subset J$ *for all $k \in \mathbf{N}$.*

As one of the consequences of the characterization of topology in $\mathcal{R}$ given in Lemma 2.2 we obtain that the classical space $\mathcal{D}$ of the smooth test functions is a subspace of $\mathcal{R}$.

Let $\mathcal{R}' = \mathcal{R}'(I)$ be the space of linear continuous functionals $\mathcal{R} \mapsto \mathbf{C}$, endowed with weak* topology, whose elements are called analogously as *distributions*. By definition of the inductive limit topology [RR64], a linear functional $f : \mathcal{R} \mapsto \mathbf{C}$ is a distribution if and only if given any $J \in \mathfrak{J}$ its restriction $f|_{\mathbb{G}^\infty(J)}$ is continuous.

As an example, given $f \in \mathbb{L}_{\mathrm{loc}}$, we may define distribution $f \in \mathcal{R}'$ whose value on the test function $\varphi \in \mathcal{R}$ is determined by the formula

$$(f, \varphi) = \int_I f(t)\varphi(t)dt.$$

Note that the map $\mathbb{L}_{\mathrm{loc}} \mapsto \mathcal{R}'$ defined above is injective, since $\mathcal{D}$ is a subspace of $\mathcal{R}$, and analogous statement is true in $\mathcal{D}'$.

Let us consider some other examples of distributions in $\mathcal{R}'$.

Example 2.3 ([DK07b])**.** Given $\tau \in I$, we define the *right* and the *left* delta-functions

$$(\delta_\tau^+, \varphi) := \varphi(\tau+), \quad (\delta_\tau^-, \varphi) := \varphi(\tau-),$$

where $\varphi \in \mathcal{R}$. In general, given $\alpha \in \mathbf{C}$, we define

$$\delta_\tau^\alpha := \alpha\delta_\tau^+ + (1-\alpha)\delta_\tau^-.$$

Clearly, if $\varphi \in \mathcal{D}$, then $(\delta_\tau^\alpha, \varphi) = \varphi(\tau)$, so $\delta_\tau^\alpha \in \mathcal{R}'$ is the extension of the classical delta-function $\delta_\tau \in \mathcal{D}'$ from $\mathcal{D}$ to $\mathcal{R}$. Let us note that together with the family of delta-functions concentrated at $\tau \in I$ there exists the family of corresponding delta-sequences: if χ_S is the characteristic function of the set S, then

$$f_k^\alpha = k\left(\alpha\chi_{(\tau, \tau+\frac{1}{2k})} + (1-\alpha)\chi_{(\tau-\frac{1}{2k}, \tau)}\right) \to \delta_\tau^\alpha$$

in $\mathcal{R}'$. Also, note that $f_k^1 - f_k^0 \to \delta_\tau^+ - \delta_\tau^-$ in $\mathcal{R}'$, while $f_k^1 - f_k^0 \to 0$ in $\mathcal{D}'$.

Example 2.4. Let $k \in \mathbf{N}_0$, $\alpha \in \mathbf{C}$. Let us define the following distributions:

$$(\delta_\tau^{(k)+}, \varphi) := (-1)^k\varphi^{(k)}(\tau+), \quad (\delta_\tau^{(k)-}, \varphi) := (-1)^k\varphi^{(k)}(\tau-), \quad \varphi \in \mathcal{R},$$

and, in general,

$$\delta_\tau^{(k)\alpha} := \alpha\delta_\tau^{(k)+} + (1-\alpha)\delta_\tau^{(k)-}, \tag{2.1}$$

which we call the *k-th derivatives of the delta-functions*. In fact, we haven't defined the derivative of a distribution yet. As it will be shown below, the distributions defined by (2.1) are indeed the derivatives of the delta-functions in $\mathcal{R}'$.

Suppose that $g \in \mathbb{G}^\infty$, $f \in \mathcal{R}'$. We define the product $gf \in \mathcal{R}'$ by the formula (see [DK07b])

$$(gf, \varphi) = (fg, \varphi) := (f, g\varphi), \quad \varphi \in \mathcal{R}, \tag{2.2}$$

where $g\varphi \in \mathcal{R}$. The operation of multiplication defined by (2.2) is associative in the sense that the equality $(gh)f = g(hf)$ holds for any g, $h \in \mathbb{G}^\infty$, $f \in \mathcal{R}'$, and, clearly, coincides with the ordinary one for the regular distributions. Furthermore, as follows from the next theorem, the operation of multiplication in $\mathcal{R}'$ is continuous.

Theorem 2.5. *Given $g_k \to g$ in $\mathbb{G}^\infty$, $f_k \to f$ in $\mathcal{R}'$, we have $g_k f_k \to gf$ in $\mathcal{R}'$.*

Example 2.6. Let us find the product of the Heaviside function $\theta_\tau \in \mathbb{G}^\infty$ and the delta-function $\delta_\tau^\alpha \in \mathcal{R}'$. According to definition (2.2) we have

$$(\theta_\tau \delta_\tau^\alpha, \varphi) = (\delta_\tau^\alpha, \theta_\tau \varphi) = \alpha\theta_\tau(\tau+)\varphi(\tau+) + (1-\alpha)\theta_\tau(\tau-)\varphi(\tau-) = \alpha\varphi(\tau+),$$

where $\varphi \in \mathcal{R}$, so

$$\theta_\tau \delta_\tau^\alpha = \alpha\delta_\tau^+.$$

Analogously, we have the following identities:

$$\theta_\tau \delta_\tau'^\alpha = \alpha\delta_\tau'^+, \quad \zeta_\tau \delta_\tau^\alpha = 0, \quad \zeta_\tau \delta_\tau'^\alpha = -\alpha\delta_\tau^+.$$

2.1. **Properties of restriction and differentiation operators.** Let

$$\Gamma : \mathcal{R}' \mapsto \mathcal{D}'$$

be the linear and, clearly, continuous operator of restriction from $\mathcal{R}$ to $\mathcal{D}$.

Theorem 2.7. *The operator Γ is surjective, that is, every distribution in $\mathcal{D}'$ has a linear continuous extension from $\mathcal{D}$ to $\mathcal{R}$.*

Proof. Indeed, since $\mathcal{R}$ is a locally-convex topological linear space, and $\mathcal{D}$ is a subspace of $\mathcal{R}$, the required extension exists by Hahn-Banach Theorem [KA82]. □

As follows from the examples provided, such extension is always non-unique (e.g., f, $f + (\delta_\tau^+ - \delta_\tau^-) \in \mathcal{R}'$ are two different extensions of the same distribution in $\mathcal{D}'$). Denote

$$\ker(\Gamma) = \{f \in \mathcal{R}' : \Gamma(f) = 0\}.$$

As follows from Example 2.3, $\ker(\Gamma) \setminus \{0\} \neq \varnothing$. Furthermore, $\ker(\Gamma)$ is a closed subspace of $\mathcal{R}'$.

In order to formulate the next theorem, which describes the structure of $\ker(\Gamma)$ and plays crucial role in our further considerations, we will need one definition and one supplementary statement.

First, given $f \in \mathcal{R}'$, we define the support $\operatorname{supp}(f) \subset I$ to be the minimal closed set such that for every $\varphi \in \mathcal{R}$ satisfying $\operatorname{supp}(\varphi) \cap \operatorname{supp}(f) = \varnothing$ we have $(f, \varphi) = 0$.

Second, given $k \in \mathbf{N}_0$, we define $\mathbb{F}_k \subset \mathbb{G}^\infty$ to be the subspace consisting of functions g such that $g^{(i)} \in \mathbb{C}$ for all $i \neq k$, $g^{(k)}$ is piece-wise continuous. Let $\mathbb{F} \subset \mathbb{G}^\infty$ be the linear space spanned by $\mathbb{F}_k$, $k \in \mathbf{N}_0$. Let $\mathcal{R}_F := \mathcal{R} \cap \mathbb{F}$ be endowed with the topology induced by the topology of $\mathcal{R}$, so that $\mathcal{R}_F$ is a subspace of $\mathcal{R}$. The following statement is technical but essential.

Lemma 2.8. *The subspace $\mathcal{R}_F$ is dense in $\mathcal{R}$. Furthermore, for every $\tau \in I$, $\varphi \in \mathcal{R}$ there exists a sequence $\{\varphi_l\}_{l=1}^\infty \subset \mathcal{R}_F$ such that $\varphi_l \to \varphi$ in $\mathcal{R}$, $\varphi_l^{(j+1)}$ is continuous in $t = \tau$ for all $j \geqslant l$, $l \in \mathbf{N}$, and*

$$|\varphi_l^{(j)}(\tau\pm) - \varphi^{(j)}(\tau\pm)| < l^{-1},$$

where $l \geqslant j$, $j \in \mathbf{N}_0$

Theorem 2.9. *1) Let $f \in \ker(\Gamma)$. Then there exist uniquely determined functions $t \mapsto c_f^k(t)$, $k \in \mathbf{N}_0$, such that for every $\varphi \in \mathcal{R}$*

$$(f, \varphi) = \sum_{k \in \mathbf{N}_0} \sum_{t \in I} c_f^k(t) \sigma_t\big(\varphi^{(k)}\big) \tag{2.3}$$

where the distribution

$$\mathcal{R} \ni \varphi \mapsto \sum_{k \in \mathbf{N}_0} \sum_{t \in I} c_f^k(t) \sigma_t\big(\varphi^{(k)}\big) \tag{2.4}$$

is defined to be the extension of the functional

$$\mathcal{R}_F \ni \varphi \mapsto \sum_{k \in \mathbf{N}_0} \sum_{t \in I} c_f^k(t) \sigma_t\big(\varphi^{(k)}\big) \tag{2.5}$$

from $\mathcal{R}_F$ to $\mathcal{R}$.

2) If $f \in \ker(\Gamma)$, $t \in I$, then $c_f^k(t) = 0$ starting with certain k.

3) If $f \in \ker(\Gamma)$, $t \in I \setminus \operatorname{supp}(f)$, then $c_f^k(t) = 0$ for all $k \in \mathbf{N}_0$.

Remark 2.10. Observe that every $\varphi \in \mathcal{R}_F$ has continuous derivatives starting with certain $K \in \mathbf{N}_0$, and the sets of points of discontinuity $T(\varphi^{(k)})$ $(0 \leqslant k \leqslant K)$ are finite, so in (2.5) there is only finite number of non-zero summands. Also, note that the extension of (2.5) exists (take f) and unique, since $\mathcal{R}_F$ is dense in $\mathcal{R}$ (Lemma 2.8).

Proof. 1) Let us show that the equality (2.3) holds for every $\varphi \in \mathcal{R}_F$. Let $\mathcal{R}_F(k,\tau)$ be the subspace of $\mathcal{R}_F$ consisting of all test functions φ such that $\varphi^{(i)} \in \mathbb{C}$ for all $i \in \mathbf{N}_0$, $i \neq k$, and $\varphi^{(k)}$ may have discontinuity only at $\tau \in I$. Let us show that for any $k \in \mathbf{N}_0$ and $\tau \in I$ we may find $a \in \mathbf{C}$ such that

$$(f, \varphi) = a\sigma_\tau(\varphi^{(k)}), \quad \varphi \in \mathcal{R}_F(k,\tau).$$

Suppose that $\varphi_0 \in \mathcal{R}_F(k,\tau)$ is such that $\sigma_\tau(\varphi_0^{(k)}) = 1$. We define:

$$a := (f, \varphi_0).$$

Let us show that the value of a does not depend on choice of φ_0. Indeed, suppose that $\varphi_1 \in \mathcal{R}_F(k,\tau)$, $\sigma_\tau(\varphi_1^{(k)}) = 1$; then $\varphi_0 - \varphi_1 \in \mathcal{D}$ and, since $f \in \ker(\Gamma)$, we obtain that $(f, \varphi_0 - \varphi_1) = 0$, i.e., $(f, \varphi_1) = a$. Now let $\varphi \in \mathcal{R}_F(k,\tau)$ be arbitrary. Then $\varphi = \sigma_\tau(\varphi)\varphi_2$ for certain $\varphi_2 \in \mathcal{R}_F(k,\tau)$ such that $\sigma_\tau(\varphi_2^{(k)}) = 1$, so due to linearity of f we have

$$(f, \varphi) = \sigma_\tau(\varphi)(f, \varphi_2) = a\sigma_\tau(\varphi).$$

Let us define:

$$c_f^k(t) := a.$$

Suppose that $\varphi \in \mathcal{R}_F$. According to Remark 2.10 there exist a number $K \in \mathbf{N}_0$ and the functions $\varphi_{ik} \in \mathcal{R}_F(k, \tau_{ik})$, where $\{\tau_{ik}\}_{i=1}^{m_k} = T(\varphi^{(k)})$, $0 \leqslant k \leqslant K$, such that

$$\varphi - \sum_{k=0}^{K} \sum_{i=1}^{m_k} \varphi_{ik} \in \mathcal{D},$$

so $\sigma_{\tau_i}(\varphi^{(k)}) = \sigma_{\tau_i}(\varphi_{ik})$, $1 \leqslant i \leqslant m_k$, $0 \leqslant k \leqslant K$. Consequently, we have

$$(f, \varphi) = \sum_{k=0}^{K} \sum_{i=1}^{m_k} (f, \varphi_{ik}) = \sum_{k=0}^{K} \sum_{i=1}^{m_k} c_f^k(\tau_i) \sigma_{\tau_i}(\varphi^{(k)}) = \sum_{k \in \mathbf{N}_0} \sum_{t \in I} c_f^k(t) \sigma_t(\varphi^{(k)}),$$

where the latter equality is obtained by adding the zero summands.

2) Suppose the contrary. Given $t \in I$, we denote $a_k = c_f^k(t)$. First, suppose that the sequence $\{a_k\}_{k=1}^\infty$ is bounded. Without loss of generality we may assume that $a_k \neq 0$, $k \in \mathbf{N}_0$. According to [Shi84] there is the test function $\varphi \in \mathcal{D}$ such that

$$|\varphi^{(k)}(t)| > |a_k|^{-1}, \quad k \in \mathbf{N}_0.$$

Let θ_t be the Heaviside function which is discontinuous at t. Clearly, $\theta_t \in \mathbb{G}^\infty$. Then $\theta\varphi \in \mathcal{R}$, and we have

$$|(\theta_t\varphi)^{(k)}(t+)| > |a_k|^{-1}, \quad (\theta_t\varphi)^{(k)}(t-) = 0, \quad k \in \mathbf{N}_0.$$

According to Lemma 2.8 there exists the sequence $\{\psi_l\}_{l=1}^\infty \subset \mathcal{R}$ such that $\psi_l^{(j+1)}$ is continuous at t for all $j \geqslant l$, $\psi_l \to \theta\varphi$ in $\mathcal{R}$, and

$$|\psi_l^{(j)}(t+) - (\theta_t\varphi)^{(j)}(t+)| < l^{-1}, \quad |\psi_l^{(j)}(t-) - (\theta_t\varphi)^{(j)}(t-)| < l^{-1}$$

for all $l \geqslant j$, $j \in \mathbf{N}_0$, so

$$|\psi_l^{(j)}(t+)| > |a_l|^{-1} - l^{-1}, \quad |\psi_l^{(j)}(t-)| < l^{-1},$$

for all $l \geqslant j$, $j \in \mathbf{N}_0$. Without loss of generality we may assume that $\psi_l^{(j)}$ is continuous on $I \setminus \{t\}$, $j \in \mathbf{N}_0$. Due to continuity of f there exists the limit

$$\lim(f, \psi_l) = (f, \theta\varphi).$$

Along with that, we have the equality

$$(f, \psi_l) = \sum_{j=0}^{l} a_j \sigma_t(\psi_l^{(j)}),$$

and

$$|(f, \psi_{l+1}) - (f, \varphi_l)| = \left|a_{l+1}\sigma_t\big(\psi_{l+1}^{(l+1)}\big)\right| > |a_{l+1}|\left(|a_{l+1}|^{-1} - 2l^{-1}\right) = 1 - 2|a_{l+1}|l^{-1} \to 1,$$

so the limit $\lim(f, \psi_l)$ does not exists, which contradicts to our assumption.

Second, suppose that the sequence $\{a_k\}_{k=1}^\infty$ is unbounded. Then we may choose the test function $\varphi \in \mathcal{D}$ such that $|\varphi^{(k)}(t)| > 1$, $k \in \mathbf{N}_0$, and repeat the previous argument. Then

$$|(f, \psi_{l+1}) - (f, \varphi_l)| = \left|a_{l+1}\sigma_t\big(\psi_{l+1}^{(l+1)}\big)\right| > |a_{l+1}|\left(1 - 2l^{-1}\right),$$

so the limit $\lim(f, \psi_l)$ does not exist, which again leads to contradiction.

3) Since $\operatorname{supp}(f)$ is a closed set, given any $k \in \mathbf{N}_0$ there exists the test function $\varphi \in \mathcal{R}_F(k, t)$ such that $\sigma_t(\varphi^{(k)}) = 1$ and $\operatorname{supp}(\varphi) \cap \operatorname{supp}(f) = \varnothing$. Then

$$c_f^k(t) = (f, \varphi),$$

but $(f, \varphi) = 0$, so $c_f^k(t) = 0$. □

Example 2.11. Suppose that in (2.3) $c_f^k(t) = 0$ for $k > 0$, $t \in I$ and for $k = 0$, $t \neq \tau$, while $c_f^0(\tau) = a$. Then

$$f = a(\delta_\tau^+ - \delta_\tau^-). \tag{2.6}$$

Suppose that in (2.3) $c_f^k(t) = 0$ if $k \neq m$, $t \in I$, and $c_f^m(t) = 1$ if $t \in I$, $m \in \mathbf{N}_0$. Then

$$(f, \varphi) = \int_I \varphi^{(m+1)}(t)dt, \quad \varphi \in \mathcal{R}. \tag{2.7}$$

Clearly, both distributions (2.6), (2.7) are in $\ker(\Gamma)$.

Let $\Delta \subset \ker(\Gamma)$ be the subspace consisting of the test functions of the form

$$\varphi \mapsto \sum_{t \in I} c(t)\sigma_t(\varphi), \quad \varphi \in \mathcal{R},$$

where $I \ni t \mapsto c(t) \in \mathbf{C}$ (equivalently, Δ consists of the elements $f \in \ker(\Gamma)$ such that $c_f^k \equiv 0$ if $k \geqslant 1$). Suppose that $\{f_k\}$ is a sequence of locally absolutely continuous functions $I \mapsto \mathbf{C}$ such that

$$f_k \to 0, \quad f_k' \to g$$

in $\mathcal{R}'$. Then it is natural to consider distribution $g \in \mathcal{R}'$ as the derivative of the zero distribution in $\mathcal{R}'$. Let us show that $g \in \Delta$. Observe first that $g \in \ker(\Gamma) \supset \Delta$, since $f_k' \to 0$ in $\mathcal{D}'$. Further, let $\varphi \in \mathcal{R}_F(0,\tau)$, $\tau \in I$. Then

$$(g,\varphi) = \lim \int_I f_k'(t)\varphi(t)dt = \lim \left(\int_a^\tau f_k'(t)\varphi(t)dt + \int_\tau^b f_k'(t)\varphi(t)dt \right) =$$

$$= \lim \left(-f_k(\tau)\sigma_\tau(\varphi) - \int_I f_k(t)\varphi'(t) \right) = -\sigma_\tau(\varphi) \lim f_k(\tau),$$

where $\lim f_k(\tau)$ exists since the value (g,φ) is defined. Let $c(t) := -\lim f_k(\tau)$. Suppose that we are given $\varphi \in \mathcal{R}_F(j,\tau)$, $j \geqslant 1$. Then

$$(g,\varphi) = \lim \int_I f_k'(t)\varphi(t)dt = -\lim \int_I f_k(t)\varphi'(t)dt = 0$$

for every $\tau \in I$. Consequently, if $\varphi \in \mathcal{R}_F$, then

$$(g,\varphi) = \sum_{t \in T(\varphi)} c(t)\sigma_t(\varphi) = \sum_{t \in I} c(t)\sigma_t(\varphi), \tag{2.8}$$

where the latter equality is obtained by adding zero summands. The distribution (2.8) can be extended uniquely to $\mathcal{R}$, by the definition $g \in \Delta$.

Conversely, let us show that given a distribution $g \in \Delta$ and a sequence of locally summable functions $\{g_k\}_{k=1}^\infty$ such that $g_k \to g$ in $\mathcal{R}'$, in assumption that the sequence of functions (i.e., primitives) $\{f_k\}_{k=1}^\infty$ defined by the formula

$$t \mapsto f_k(t) := \int_{t_0}^t g_k(s)ds \quad (t_0 \in I)$$

tends to a regular distribution $f \in \mathcal{R}'$, we have that f, as an ordinary function, is identically equal to a constant. If we show that there exists $r \in \mathbf{C}$ such that

$$(f_k,\varphi) \to r \int_I \varphi(t)dt,$$

for every $\varphi \in \mathcal{R}$, then the proof would be complete. Indeed, given $\varphi \in \mathcal{R}$ (suppose that $\mathrm{supp}(\varphi) \subset [u,v] \subset I$), we have

$$\int_I f_k(t)\varphi(t)dt = \int_u^v \left(\int_{t_0}^t g_k(s)ds \right) d \int_u^t \varphi(s)ds =$$

$$= \left(\int_{t_0}^v g_k(s)ds \right) \int_u^v \varphi(s)ds - \int_u^v \left(\int_u^t \varphi(s)ds \right) g_k(t)dt \to$$

$$\to \left(g, \chi_{(t_0,v)}\right) \int_u^v \varphi(s)ds - \left(g, \int_u^t \varphi(s)ds \chi_{(u,v)}(t) \right) =$$

$$= \left(c_g(v) - c_g(t_0)\right) \int_I \varphi(s)ds - c_g(v) \int_I \varphi(s)ds = -c_g(t_0) \int_I \varphi(s)ds,$$

so we may put $r := -c_g(t_0)$. The proof is complete.

Taking into account the above considerations, we define the derivative of the distribution $f \in \mathcal{R}'$ to be any distribution $f' \in \mathcal{R}'$ such that

$$f' \in \frac{df}{dt} + \Delta, \text{ where } \left(\frac{df}{dt}, \varphi\right) := -(f, \varphi'), \quad \varphi \in \mathcal{R}, \tag{2.9}$$

where φ' is assumed to be defined almost everywhere. We denote by $D(f) \subset \mathcal{R}'$ the family of all derivatives of the distribution $f \in \mathcal{R}'$. As it immediately follows from the definition, the operator of differentiation $f \mapsto D(f)$ is multi-valued.

We define the derivatives of higher orders inductively.

Example 2.12. Let θ_τ be Heaviside function. Then

$$\left(\frac{d\theta_\tau}{dt}, \varphi\right) = -\int_\tau^b \varphi'(t)dt = \varphi(\tau+),$$

so $(\frac{d\theta_\tau}{dt}, \varphi) = \delta_\tau^+$. Consequently, for every $\alpha \in \mathbf{C}$ delta-function δ_τ^α is the derivative of θ_τ.

Example 2.13. As follows from definition (2.9) the distribution (2.1) is indeed the k-th derivative of δ_τ^α. Furthermore,

$$\delta_\tau'^\alpha + \beta(\delta_\xi^+ - \delta_\xi^-)$$

is the derivative of δ_τ^α for any $\beta \in \mathbf{C}$, $\xi \in I$,

$$\delta_\tau'^\alpha + \beta(\delta_\xi'^+ - \delta_\xi'^-)$$

is the derivative of $\delta_\tau^\alpha + \beta(\delta_\xi^+ - \delta_\xi^-)$, and

$$\delta_\tau^\alpha + \beta(\delta_\xi^+ - \delta_\xi^-)$$

is the derivative of θ_τ for any $\beta \in \mathbf{C}$, $\xi \in I$.

Example 2.14. Let $f : I \mapsto \mathbf{C}$ be a locally absolutely continuous function. Then

$$\begin{aligned}(f', \varphi) &= \int_I f'(t)\varphi(t)dt = \int_I \varphi(t)df(t) = \\ &= -\int_I f(t)d\varphi(t) = -\int_I f(t)\varphi'(t)dt - \sum_{t \in I} f(t)\sigma_t(\varphi), \quad \varphi \in \mathcal{R}\end{aligned} \tag{2.10}$$

where, as usual, the latter sum is defined to be the extension of the corresponding functional from $\mathcal{R}_F$ to $\mathcal{R}$. Since $\sum_{t \in I} f(t)\sigma_t(\varphi) \in \Delta$, we have

$$f' + \Delta = D(f),$$

where f' is defined by (2.10).

Theorem 2.15. *Suppose that $f \in \mathcal{R}'$, and we are given its m-th derivative $f^{(m)} \in \mathcal{R}'$. Then the intermediate derivatives $f^{(k)}$, $0 \leqslant k \leqslant m-1$, are determined uniquely.*

Proof. We will prove this result by induction over k. By definition $f^{(0)} = f$ is determined uniquely. Suppose that $f^{(i)}$ are uniquely determined for $0 \leqslant i \leqslant k-1$. Then, since the primitive of a distribution is defined uniquely up to a constant summand, we have that $f^{(k)}$ is defined uniquely by $f^{(m)}$ up to a polynomial of degree $m-k-1$. Along with that, if $f_1^{(k)}$, $f_2^{(k)}$ are two different derivatives of $f^{(k-1)}$, then

$$f_1^{(k)} - f_2^{(k)} \in \Delta.$$

Now since Δ does not contain polynomials except the one identically equal to zero, we have that $f^{(k)}$ is uniquely determined. □

2.2. **Theorems on structure of distributions.** The following statement is a generalization of well-known result on structure of distributions in $\mathcal{D}'$ [Shi84].

Theorem 2.16. *Let $f \in \mathcal{R}'$. Then there exist functions $f_k \in \mathbb{L}_{\mathrm{loc}}$ and functions $c^k : I \mapsto \mathbf{C}$, $k \in \mathbf{N}_0$, such that*

$$(f,\varphi) = \sum_{k\in\mathbf{N}_0} (-1)^k \int_I f_k(t)\varphi^{(k)}(t)dt + \sum_{k\in\mathbf{N}_0}\sum_{t\in I} c^k(t)\sigma_t(\varphi^{(k)}), \quad \varphi \in \mathcal{R}. \tag{2.11}$$

If $f = 0$ for $t < t_0$, that is, for every test function $\varphi \in \mathcal{R}$ such that $\operatorname{supp}(\varphi) \subset (a, t_0)$ *we have $(f,\varphi) = 0$, then there exist f_k such that $f_k = 0$ for $t < t_0$.*

Proof. According to [Shi84] there exist functions $f_k \in \mathbb{L}_{\mathrm{loc}}$, $k \in \mathbf{N}_0$, such that

$$(f,\varphi) = \sum_{k\in\mathbf{N}_0} (-1)^k \int_I f_k(t)\varphi^{(k)}(t)dt, \quad \varphi \in \mathcal{D}. \tag{2.12}$$

(if $f = 0$ for $t < t_0$, then there exist f_k such that $f_k = 0$ for $t < t_0$). If we consider in (2.12) the test functions $\varphi \in \mathcal{R}$, then we get certain extension of $\Gamma(f)$ from $\mathcal{D}$ to $\mathcal{R}$. According to Theorem 2.9 the family of all extensions of $\Gamma(f)$ from $\mathcal{D}$ to $\mathcal{R}$ consists of all distributions of form (2.11). Since f is one of such extensions, there exist locally-summable functions c^k such that (2.11) is true. □

Theorem 2.17. *Let $f \in \mathcal{R}'$,* $\operatorname{supp}(f) \subset \{\tau\}$. *Then*

$$f = \sum_{k=0}^{K} a_k \delta_\tau^{(k)\alpha_k} + \sum_{k=0}^{M} b_k \left(\delta_\tau^{(k)+} - \delta_\tau^{(k)-}\right) \tag{2.13}$$

for certain K, $M \in \mathbf{N}_0$, a_k, $b_k \in \mathbf{C}$, $\alpha_k \in \mathbf{C}$.

As it follows from Theorem 2.17, in $\mathcal{R}'$ there are no other extensions of delta-function and its derivatives from $\mathcal{D}$ to $\mathcal{R}$, concentrated at a single point, except defined in the examples above.

Proof. If $\operatorname{supp}(f) = \varnothing$, then $f = 0$, and the proof is complete. So, we may suppose that $\operatorname{supp}(f) = \{\tau\}$. As follows from the definition of support, we have the inclusion $\operatorname{supp}\big(\Gamma(f)\big) \subset \{\tau\}$. According to [Shi84] there exist $K \in \mathbf{N}_0$, $a_k \in \mathbf{C}$, $0 \leqslant k \leqslant K$, such that

$$\Gamma(f) = \sum_{k=0}^{K} a_k \delta_\tau^{(k)}.$$

Let $\alpha_k \in \mathbf{C}$, $0 \leqslant k \leqslant K$. Then

$$\varphi \mapsto \sum_{k=0}^{K} a_k \delta_\tau^{(k)\alpha_k}$$

is an extension of $\Gamma(f)$ from $\mathcal{D}$ to $\mathcal{R}$. According to Theorem 2.9 any extension of $\Gamma(f)$, concentrated at τ, has form (2.13) for certain $M \in \mathbf{N}_0$, a_k, $b_k \in \mathbf{C}$, $\alpha_k \in \mathbf{C}$. Since f is one of such extensions, we obtain the statement of the theorem. □

Let us show that the space of distributions $\mathcal{R}' = \mathcal{R}'(I)$ is isomorphic (as a topological linear space) to the space of distributions with the smooth test functions defined on a certain totally disconnected set. Namely, let I_* be the maximal ideal space of the Banach algebra $\mathbb{G} = \mathbb{G}(I)$, that is, the space of all continuous algebra homomorphisms $\mathbb{G} \mapsto \mathbf{C}$, endowed with weak* topology [Gam69]. As is shown in [BK07], I_* is a totally disconnected Hausdorff space which consists of homomorphisms of the form

$$g \mapsto g(t-), \quad g \mapsto g(t+),$$

where $g \in \mathbb{G}$, $t \in I$. We denote these homomorphisms by $t-$ and $t+$, respectively, and use notations $g(t-)$ for $(t-)(g)$ and $g(t+)$ for $(t+)(g)$, $g \in \mathbb{G}$. So, as the set, I_* is in one-to-one correspondence with $I \times \{-1, 1\}$, where -1 stands for the left hand-side limit evaluation homomorphism, and 1 stands for the right hand-side limit evaluation homomorphism. In what follows, we put $[t+] = [t-] := t$. Since algebra $\mathbb{G}$ is regular and symmetric, i.e.,

$$\|g^2\|_{\mathbb{L}_\infty} = \|g\|^2_{\mathbb{L}_\infty}$$

and $\bar{g} \in \mathbb{G}$ for every $g \in \mathbb{G}$, we have that $\mathbb{G}$ is isometrically isomorphic to $\mathbb{C}(I_*)$ [Gam69]. Let $\mathbb{C}^\infty(I_*)$ be the algebra consisting of infinitely continuously differentiable functions $g \in \mathbb{C}(I_*)$, that is, the functions such that for every $k \in \mathbf{N}$, $t\cdot \in I_*$ there exists the limit

$$g^{(k)}(t\cdot) = \lim_{r \to t\cdot,\ r \neq t\cdot} \left(\frac{g^{(k-1)}(r) - g^{(k-1)}(t\cdot)}{[r] - t} \right)$$

and $g^{(k)} \in \mathbb{C}(I_*)$ (by definition, $g^{(0)} := g$). We introduce in $\mathbb{C}^\infty(I_*)$ the countable family of norms

$$\|g\|_k = \max_{0 \leqslant i \leqslant k} \sup_{t\cdot \in I_*} |g^{(i)}(t\cdot)|, \quad g \in \mathbb{C}^\infty(I_*), \quad k \in \mathbf{N}_0.$$

Lemma 2.18. *$\mathbb{G}^\infty(I)$ is isometrically isomorphic to $\mathbb{C}^\infty(I_*)$.*

Proof. Suppose that $f \in \mathbb{G}(I)$. In what follows, we denote the image of f in $\mathbb{C}^\infty(I_*)$ under the map defined above by f_*. Let $g \in \mathbb{G}^\infty(I)$. Show that for every $k \in \mathbf{N}_0$

$$(g^{(k)})_* = (g_*)^{(k)}. \tag{2.14}$$

Let us prove this statement inductively by k. Clearly,

$$(g^{(0)})_* = (g_*)^{(0)}.$$

Suppose that $m \in \mathbf{N}$, and (2.14) is true for $k = m - 1$. We have to show now that (2.14) is true for $k = m$. Observe that due to the fact that the set of points of discontinuity $T(g^{(m)})$ has zero measure, we have the equality

$$g^{(m)}(t\pm) = \operatorname*{esslim}_{s \to t\pm,\ s \neq t} \left(\frac{g^{(m-1)}(s) - g^{(m-1)}(t\pm)}{s - t} \right) = \lim_{s \to t\pm,\ s \neq t} \left(\frac{g^{(m-1)}(s\cdot) - g^{(m-1)}(t\pm)}{s - t} \right)$$

As it follows from our assumption and from the definition of the topology in I_*, the latter is equal to

$$\lim_{r \to t\pm,\ r \neq t} \left(\frac{g_*^{(m-1)}(r) - g_*^{(m-1)}(t\pm)}{[r] - t} \right) = g_*^{(m)}(t\pm)$$

for every $t \in I$, so the equality (2.14) holds for $k = m$, so $\mathbb{G}^\infty(I)$ is isomorphic to $\mathbb{C}^\infty(I_*)$. Further, as follows from (2.14), we have that

$$\|g\|_k = \|g_*\|_k, \quad k \in \mathbf{N}_0,$$

so $\mathbb{G}^\infty(I)$ is isometrically isomorphic to $\mathbb{C}^\infty(I_*)$. □

Let $\mathcal{D}(I_*)$ be the space consisting of elements $\varphi \in \mathbb{C}^\infty(I_*)$ such that there exist c_φ, $d_\varphi \in I$ possessing the property $\varphi(t\cdot) = 0$ if $t < c_\varphi$, $\varphi(t\cdot) = 0$ if $t > d_\varphi$, endowed with the convergence: $\varphi_k \to \varphi$ in $\mathcal{D}(I_*)$ if and only if $\varphi_k \to \varphi$ in $\mathbb{C}^\infty(I_*)$ and there exist $c, d \in I$ such that $\varphi_k(t\cdot) = 0$ if $t < c_\varphi$, $\varphi_k(t\cdot) = 0$ if $t > d_\varphi$ for all $k \in \mathbf{N}$. Let $\mathcal{D}'(I_*)$ be the space of all continuous linear functionals $\mathcal{D}(I_*) \mapsto \mathbf{C}$. As a simple corollary of the results above, we obtain the following statement.

Theorem 2.19. *$\mathcal{R}'(I)$ is isomorphic to $\mathcal{D}'(I_*)$.*

3. Linear differential equations

Let $\mathcal{R}^{n\prime}$ be the space of vector-valued distributions $\mathcal{R} \mapsto \mathbf{C}^n$ with the operations and convergence defined componentwise. We introduce analogous notations for the spaces of vector-valued and matrix-valued distributions and functions $I \mapsto \mathbf{C}^n$ and $I \mapsto \mathbf{C}^{n\times m}$, respectively.

Let us consider in the space $\mathcal{R}^{n\prime}$ the following linear differential equation

$$x' - A(t)x = 0 \tag{3.1}$$

with the matrix of coefficients $A \in \mathbb{G}^\infty_{n\times n}$. Analogously, the solution of the differential equation (3.1) is the distribution $x \in \mathcal{R}^{n\prime}$ which possesses a derivative $x' \in \mathcal{R}^{n\prime}$ such that after the substitution of x and x' in (3.1) we obtain the identity $(x', \varphi) = (A(t)x, \varphi)$ for all $\varphi \in \mathcal{R}$.

Theorem 3.1. *There are no other solutions in the space $\mathcal{R}^{n\prime}$ of the differential equation* (3.1) *except the classical ones.*

Proof. Without loss of generality we conduct our proof for the case $n = 1$. First, let us show that there are no other solutions of equation (3.1) being considered on the space of smooth test functions $\mathcal{D}$ except the classical ones. Let $x \in \mathcal{R}'$ be a solution of equation (3.1). By definition, the derivatives of x coincide on $\mathcal{D} \subset \mathcal{R}$, so $(x', \varphi) = -(x, \varphi')$. Also note that if we are given a distribution $h \in \mathcal{R}'$ and a function $g \in \mathbb{G}^\infty$ which is continuous (and, thus, differentiable), then we have $((gh)', \varphi) = -(gh, \varphi') = -(h, g\varphi') = (h, -(g\varphi)' + g'\varphi)$ for all $\varphi \in \mathcal{D}$, that is, we have the identity $(gh)' = h'g + hg'$ on $\mathcal{D}$. Now let us represent x in the form $x = e^B y$, where $B \in \mathbb{G}^\infty$ is a (continuous) primitive of A, and $y \in \mathcal{R}'$. Then

$$x' = (e^B)'y + e^B y',$$

on $\mathcal{D}$, so our equation is equivalent to

$$Ae^B y + e^B y' = Ae^B y,$$

which is, in turn, implies that $(e^B y', \varphi) = 0$ for all $\varphi \in \mathcal{D}$. We may multiply both parts of this equality by e^{-By} to get the identity $(y', \varphi) = 0$ for all $\varphi \in \mathcal{D}$. According to [Shi84] $y \equiv \text{const}$. Thus, x restricted to $\mathcal{D}$ can be only an ordinary function (that is, regular distribution).

We obtained that if x is the solution of (3.1) in $\mathcal{R}'$, then

$$x = x_0 + y,$$

where x_0 is the classical solution, $y \in \ker(\Gamma)$. Clearly, y is also a solution of equation (3.1), i.e.,

$$\sum_{k\in\mathbf{N}_0} \sum_{t\in I} c^k_y(t)\sigma_t(\varphi^{(k+1)}) + (z, \varphi) = \sum_{k\in\mathbf{N}_0} \sum_{t\in I} c^k_y(t)\sigma_t\left((A\varphi)^{(k)}\right), \quad \varphi \in \mathcal{R},$$

where $z \in \Delta$. Let us choose certain $\tau \in I$, $\varphi \in \mathcal{R}(k, \tau)$. Then, since $c^k_y(\tau) = 0$ starting with certain K according to Theorem 2.9, we have the equality

$$\sum_{k=0}^{K} c^k_y(\tau)\sigma_\tau(\varphi^{(k+1)}) + c_z(\tau)\sigma_\tau(\varphi) = \sum_{k=0}^{K} c^k_y(\tau)\sigma_\tau\left((A\varphi)^{(k)}\right). \tag{3.2}$$

Since $\sigma(\varphi^{(K+1)})$ is not contained in the right-hand side of (3.2), we find that $c^K_y(\tau) = 0$. Consequently, the right-hand side of (3.2) does not contain the summand corresponding to $k = K$. Then $\sigma_\tau(\varphi^{(K)})$ is not contained in the right-hand side of (3.2), so $c^{K-1}_y(\tau) = 0$. We may continue this process to obtain that $c^1_y(\tau) = 0$. Then the right-hand side of (3.2) does not contain $\sigma_\tau(\varphi')$ and, as a result, we get $c^0_y(\tau) = 0$ (and also $c_z(\tau) = 0$). Now since $\tau \in I$ was chosen arbitrarily, we obtain that $c^k_y(t) = 0$ for all $t \in I$, $k \in \mathbf{N}_0$. Consequently, the only solution of differential equation (3.1) in $\ker(\Gamma)$ is the zero solution, so $x = x_0$. $\square$

Further, let us consider in the space $\mathcal{R}'$ the differential equation

$$x' = f, \tag{3.3}$$

where $f \in \mathcal{R}'$. In what follows, we assume that $f = 0$ if $t < t_0$. Analogously, we call the solution of the differential equation (3.3) the distribution $x \in \mathcal{R}'$ which possesses a derivative $x' \in \mathcal{R}'$ such that after substitution of x' in (3.3) the equation (3.3) becomes the identity $(x', \varphi) = (f, \varphi)$, $\varphi \in \mathcal{R}$.

Theorem 3.2. *There exists solution x of equation* (3.3) *such that $x = 0$ if $t < t_0$.*

The solution x of equation (3.3) is called the *primitive* of distribution f.

Proof. According to Theorem 2.16 there exist the locally-summable functions f_k which are equal to 0 for $t < t_0$, and functions c^k, $k \in \mathbf{N}_0$ such that (2.11) holds. Let us define (let $t_1 < t_0$, $t_1 \in I$):

$$\begin{aligned}(x, \varphi) := -\int_I \left(\int_{t_1}^{t} f_0(s)ds \right) \varphi'(t)dt + \\ + \sum_{k \in \mathbf{N}_0} (-1)^k \int_I f_{k+1}(t)\varphi^{(k)}(t)dt - \sum_{k \in \mathbf{N}_0} \sum_{t \in I} c^{k+1}(t)\sigma_t(\varphi^{(k)}), \quad \varphi \in \mathcal{R}.\end{aligned} \tag{3.4}$$

Now we may define the derivative

$$\begin{aligned}(x', \varphi) = \int_I f_0(t)\varphi(t)dt + \sum_{k \in \mathbf{N}_0} (-1)^{k+1} \int_I f_{k+1}(t)\varphi^{(k+1)}(t)dt + \\ + \sum_{k \in \mathbf{N}_0} \sum_{t \in I} c^{k+1}(t)\sigma_t(\varphi^{(k+1)}) + z,\end{aligned} \tag{3.5}$$

where $z \in \Delta$,

$$z = \sum_{t \in I} c^0(t)\sigma_t(\varphi) + \sum_{t \in I} \left(\int_{t_1}^{t} f_0(s)ds \right) \sigma_t(\varphi), \quad \varphi \in \mathcal{R}. \tag{3.6}$$

As follows from Example 2.14, the distribution defined by (3.5), (3.6) is indeed in $D(x)$, and $x' = f$. From (3.4) we have that $x = 0$ for $t < t_0$, and the proof is complete. $\square$

Let us consider in the space $\mathcal{R}^{n\prime}$ the linear differential equation of the general form

$$x' - A(t)x = f \tag{3.7}$$

where $A \in \mathbb{G}^\infty_{n \times n}$, $f \in \mathcal{R}^{n\prime}$. The linear systems of form (3.7) were considered in [Shi84] in the classical space of distributions $\mathcal{D}'$ in assumption that $A \in \mathbb{C}^\infty_{n \times n}$.

We call the solution of the differential equation (3.7) the distribution $x \in \mathcal{R}^{n\prime}$ which possesses the derivative $x' \in \mathcal{R}^{n\prime}$ such that after substitution of x and x' in (3.7) equation (3.7) becomes the identity

$$(x' - A(t)x, \varphi) = (f, \varphi), \quad \varphi \in \mathcal{R}.$$

Let us note that the left-hand side of differential equation (3.7) contains the product of a function from $\mathbb{G}^\infty_{n \times n}$ (generally discontinuous) and a distribution, which is correctly defined in the space $\mathcal{R}'$, but in general is undefined in the classical space $\mathcal{D}'$.

Let $t_0 \in I$. In what follows, we suppose that $f = 0$ if $t < t_0$. Let us consider Cauchy problem for the differential equation (3.7) with the initial condition

$$x = 0 \text{ if } t < t_0 \tag{3.8}$$

(we will consider Cauchy problem with the initial condition of the general form below).

Theorem 3.3. *There exists the unique solution x of Cauchy problem* (3.7), (3.8). *Furthermore, $x = Xy$, where X is the fundamental solution of the corresponding homogeneous system, y is the primitive of the distribution $X^{-1}f$ equal to* 0 *if $t < t_0$.*

Proof. Without loss of generality we conduct our proof for the case $n = 1$. If X is the fundamental solution of the corresponding homogeneous system, then, clearly, $X \in \mathbb{G}^\infty$ and there exists $X^{-1} \in \mathbb{G}^\infty$, so the product $X^{-1}f \in \mathcal{R}'$. According to Theorem 3.2 there exists the primitive of $X^{-1}f$ which is equal to zero if $t < t_0$. Let us denote this primitive by y. Now let us determine one of the derivatives of y by the equality

$$(y', \varphi) := (X^{-1}f, \varphi), \quad \varphi \in \mathcal{R}.$$

We multiply both sides of $y' = X^{-1}f$ by X:

$$(Xy', \varphi) = (f, \varphi),$$

so $Xy' = f$. Let us note that according to the definition of the derivative we have $y' = \frac{dy}{dt} + z$, where $z \in \Delta$. Let us show that $Xz \in \Delta$. Indeed, since X is continuous, we have that if

$$(z, \varphi) = \sum_{t \in I} c_z(t)\sigma_t(\varphi),$$

then

$$(Xz, \varphi) = (z, X\varphi) = \sum_{t \in I} c_z(t)X(t)\sigma_t(\varphi),$$

i.e., $c_{Xz}(t) := c_z(t)X(t)$, $t \in I$. Further, let us choose a particular derivative of the distribution Xy. We have:

$$\left(\frac{d}{dt}(Xy), \varphi\right) = -(Xy, \varphi') = -(y, X\varphi') = -(y, (Xy)' - X'\varphi) = \left(X\frac{dy}{dt}, \varphi\right) + (X'y, \varphi),$$

where $X' \in \mathbb{G}^\infty$. Since $X\frac{dy}{dt} = f - Xz$, and $Xz \in \Delta$, we may define the derivative

$$(Xy)' := \frac{d}{dt}(Xy) + Xz.$$

Then

$$(Xy)' = f + AXy,$$

so $x = Xy$ is the solution of the differential equation (3.7). Further, since $y = 0$ for $t < t_0$, then $x = 0$ for $t < t_0$: if$\varphi \in \mathcal{R}$, $\mathrm{supp}(\varphi) \subset (a, t_0)$, then $X\varphi \in \mathcal{R}$, $\mathrm{supp}(X\varphi) \subset (a, t_0)$ and

$$(x, \varphi) = (Xy, \varphi) = (y, X\varphi) = 0.$$

Consequently, x is the solution of the Cauchy problem (3.7), (3.8).

Finally, let us show that x defined above is the only solution of the Cauchy problem. Indeed, if x_1 and x_2 are two solutions of (3.7), (3.8), then $x_1 - x_2$ is the solution of Cauchy problem (3.1), (3.8). According to Theorem 3.1 the difference $x_1 - x_2$ is the ordinary solution of system (3.1), which is identically equal to zero according to classical uniqueness theorem due to condition (3.8), so $x_1 = x_2$. □

Now let us consider Cauchy problem for the differential equation (3.7) with the initial condition of the general form

$$x = x_0 \text{ if } t < t_0, \tag{3.9}$$

where $x_0 \in \mathbf{C}^n$. Consider the following differential equation:

$$y' - A(t)y = f + x_0\delta^\alpha_{t_0}, \tag{3.10}$$

where $\delta^\alpha_{t_0} \in \mathcal{R}$, $\alpha \in \mathbf{C}$ is arbitrary. Clearly, we have $f + x_0\delta^\alpha_{t_0} \in \mathcal{R}^{n\prime}$, so by Theorem 3.3 the solution y of Cauchy problem (3.10), (3.8) exists in $\mathcal{R}^{n\prime}$ and unique. We call y the solution of Cauchy problem (3.7), (3.9).

The next statement shows that this definition agrees with the classical one.

Theorem 3.4. *The following statements are true:*

1) The solution of the Cauchy problem (3.7), (3.9) does not depend on choice of α in (3.10).

2) If f is a regular distribution, then the solution of Cauchy problem (3.7), (3.9) coincides for $t > t_0$ with the solution of the corresponding ordinary Cauchy problem

$$x' - A(t)x = f(t), \quad t > t_0, \quad x(t_0+) = x_0.$$

Proof. Let y be the solution of Cauchy problem (3.10), (3.8) (i.e., the solution of Cauchy problem (3.7), (3.9) according to our definition). Evidently, if y_1 is the solution of Cauchy problem (3.7), (3.8), and y_2 is the solution of Cauchy problem for the differential equation

$$y_2' - A(t)y_2 = x_0\delta_{t_0}^\alpha$$

with initial condition (3.8), then by uniqueness of solution $y = y_1 + y_2$. Note that y_1 does not depend on value of $\alpha \in \mathbf{C}$, so in order to complete the proof it suffices to show that y_2 does not depend on α. By Theorem 3.3 $y_2 = Xy_3$, where X is the fundamental solution of the corresponding homogeneous system, and y_3 is the primitive of distribution $X^{-1}x_0\delta_{t_0}^\alpha$. Clearly, X^{-1} is continuous on I, hence (without loss of generality, $n = 1$)

$$(X^{-1}x_0\delta_{t_0}^\alpha, \varphi) = (\delta_{t_0}^\alpha, X^{-1}x_0\varphi) = X^{-1}(t_0)x_0\varphi(t_0),$$

and the proof of the first statement is complete.

In order to prove the second statement, first observe that given regular f, the solution x of Cauchy problem (3.7), (3.9) is a regular distribution. Second, since $x_0\delta_{t_0}^\alpha$ coincides with the zero distribution on $I\backslash\{t_0\}$, x coincides with solutions x_1, x_2 of differential equation $x' = A(t)x+f(t)$ on (a, t_0) and (t_0, b), respectively. By definition, $x = x_1 = 0$ on (a, t_0). Further, since A is in $\mathbb{G}_{n\times n}^\infty$ and, thus, locally-summable on I together with f, there exists the limit $x(t_0+)$. Finally, since for every α delta-function $\delta_{t_0}^\alpha$ is the derivative in $\mathcal{R}'$ of Heaviside function θ_{t_0} discontinuous at t_0, we have that

$$x(t_0+) - x(t_0-) = x_0\big(\theta_{t_0}(t_0+) - \theta_{t_0}(t_0-)\big),$$

so $x(t_0+) = x_0$. □

Theorem 3.5. *The solution of the Cauchy problem (3.7), (3.9) depends continuously on $f \in \mathcal{R}^{n\prime}$ and $x_0 \in \mathbf{C}^n$.*

Proof. By definition, the solution of Cauchy problem (3.7), (3.9) is the solution of Cauchy problem (3.10), (3.8). Note that the continuous dependence of the solution on the initial value x_0 will follow from the continuous dependence on f, so we may restrict ourselves to the proof of the first statement. Now, as follows from the decomposition obtained in the proofs of Theorem 3.2 (see (3.4), (3.5)), given distribution in $\mathcal{R}'$, its primitive, equal to 0 for $t < t_0$, depends continuously on this distribution. Thus, since all operations in $\mathcal{R}'$ which arise in construction of the solution of the Cauchy problem (3.7), (3.8) (in particular, (3.10), (3.8)) in Theorem 3.3 are continuous (including the operation of multiplication, which is continuous by Theorem 2.5), the solution depends continuously on $f \in \mathcal{R}'$. □

3.1. **Linear differential equations of higher orders.** Let us consider in $\mathcal{R}^{n\times n\prime}$ the following linear differential equation of order m:

$$X^{(m)} - A_{m-1}X^{(m-1)} - \cdots - A_0X = F, \tag{3.11}$$

where $A_i \in \mathbb{G}_{n\times n}^\infty$, $(0 \leqslant i \leqslant m)$, $F \in \mathcal{R}^{n\times n\prime}$.

The solution of differential equation (3.11) is the distribution $X \in \mathcal{R}^{n\times n\prime}$ which possesses its m-th derivative $X^{(m)}$ such that after substitution of X, X', …, $X^{(m)}$ in (3.11) we obtain the equality

$$(X^{(m)} - A_{m-1}X^{(m-1)} - \cdots - A_0X, \varphi) = (F, \varphi)$$

for all $\varphi \in \mathcal{R}$. This definition is correct according to Theorem 2.15: it is sufficient to assume that only X and $X^{(m)}$ are specified in order to have all intermediate derivatives $X^{(k)}$, $1 \leqslant k \leqslant m-1$, uniquely determined.

First, let us consider the Cauchy problem for equation (3.11) with the homogeneous initial conditions

$$X^{(k)} = 0, \quad t < t_0, \quad 0 \leqslant k \leqslant m-1, \tag{3.12}$$

where $t_0 \in I$.

Theorem 3.3'. There exists the unique solution X of Cauchy problem (3.11), (3.12). Furthermore, $X = \sum_{i=1}^m X_i T_i$, where $\{X_i\}_{i=1}^m$ is the fundamental system of solutions of the corresponding homogeneous differential equation, T_i is the primitive of distribution $Z_i F$ equal to 0 if $t < t_0$, and Z_i is the i-th element of the first column of R^{-1}, where

$$R = \begin{pmatrix} X_1 & \ldots & X_m \\ & \ldots & \\ X_1^{(m-1)} & \ldots & X_m^{(m-1)} \end{pmatrix}.$$

Proof. Following standard scheme, we can make a substitution $Y_k = X^{(k-1)}$, $1 \leqslant k \leqslant m$, to reduce differential equation (3.11) to the linear system of the form (3.7),

$$\begin{pmatrix} Y_m' \\ Y_{m-1}' \\ \ldots \\ Y_1' \end{pmatrix} = \begin{pmatrix} A_{m-1} & A_{m-2} & \ldots & A_1 & -A_0 \\ 1 & 0 & \ldots & 0 & 0 \\ & & \ldots & & \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix} \begin{pmatrix} Y_m \\ Y_{m-1} \\ \ldots \\ Y_1 \end{pmatrix} + \begin{pmatrix} F \\ 0 \\ \ldots \\ 0 \end{pmatrix} \tag{3.13}$$

As it immediately follows from the definition, every solution of differential equation (3.11) is the solution of system (3.13) (see above), and vice versa. The homogeneous initial conditions (3.12), being rewritten for system (3.13), coincide with the homogeneous initial conditions

$$Y_k = 0, \quad t < t_0, \quad 1 \leqslant k \leqslant m, \tag{3.14}$$

so it suffices to apply Theorem 3.3 to prove the existence and uniqueness of the solution of Cauchy problem (3.11), (3.12). Further, if $\{X_i\}_{i=1}^m$ is the fundamental system of solutions of the corresponding homogeneous linear differential equation, then R is the fundamental matrix of the homogeneous system corresponding to (3.13). According to Theorem 3.3 the solution of Cauchy problem (3.13), (3.14) admits representation in the form of the product RT, where $T = (T_i)_{i=1}^m$ is the primitive of the distribution $R^{-1}(F, 0, \ldots, 0)^\top$ which is equal to 0 for $t < t_0$. Since $X = Y_1$, we have that $X = \sum_{i=1}^m X_i T_i$. The representation of T_i is straightforward. The proof is complete. □

Now consider Cauchy problem for the differential equation (3.11) with the initial conditions of the general form, that is,

$$X^{(k)} = X_k, \quad t < t_0, \quad 0 \leqslant k \leqslant m-1, \tag{3.15}$$

where $X_k \in \mathbf{C}^{n \times n}$. We need to define the solution of Cauchy problem (3.11), (3.15). For this purpose we consider the Cauchy problem for the differential equation of the form (3.11),

$$\sum_{k=0}^{m} A_k \left(Z^{(k)} - \sum_{j=0}^{k-1} X_j \delta_{t_0}^{(k-j-1)\alpha} \right) = F, \tag{3.16}$$

where $\alpha \in \mathbf{C}$ is arbitrary, A_m is defined to be the identity matrix, with the homogeneous initial conditions

$$Z^{(k)} = 0, \quad t < t_0, \quad 0 \leqslant k \leqslant m-1, \tag{3.17}$$

that was already considered above. So, if Z is the solution of problem (3.16), (3.17), which always exists and unique in virtue of Theorem 3.3', then we call Z the solution of the Cauchy problem (3.11), (3.15). The following result shows that this definition is quite natural.

Theorem 3.4'. The following statements are true:

1) The solution of the Cauchy problem (3.11), (3.15) does not depend on α in (3.16).

2) If F is a regular distribution, then the solution of the Cauchy problem (3.11), (3.15) coincides for $t > t_0$ with the solution of the corresponding ordinary Cauchy problem

$$X^{(m)} - A_{m-1}X^{(m-1)} - \dots - A_0X = F, \quad X^{(k)}(t_0+) = X_k, \quad 0 \leqslant k \leqslant m-1.$$

Proof. Let us consider Cauchy problem for the linear system

$$\begin{pmatrix} Y_m' \\ Y_{m-1}' \\ \dots \\ Y_1' \end{pmatrix} = \begin{pmatrix} A_{m-1} & \dots & A_1 & A_0 \\ 1 & \dots & 0 & 0 \\ & \dots & & \\ 0 & \dots & 1 & 0 \end{pmatrix} \begin{pmatrix} Y_m \\ Y_{m-1} \\ \dots \\ Y_1 \end{pmatrix} + \begin{pmatrix} F + X_{m-1}\delta_{t_0}^{\alpha} \\ X_{m-2}\delta_{t_0}^{\alpha} \\ \dots \\ X_0\delta_{t_0}^{\alpha} \end{pmatrix}, \tag{3.18}$$

with the initial conditions as in (3.14). As it follows from Theorems 3.3 and 3.3', the solutions of Cauchy problems (3.18), (3.14) and (3.11), (3.15), respectively, exist and unique. So, in order to prove this theorem we have to show that $Y_1 = Z$, the rest will follow from Theorem 3.4 being applied to system (3.18). Suppose that Z is the solution of (3.11), (3.15). Let us put

$$Y_k = Z^{(k-1)} - \sum_{j=0}^{k-2} X_j \delta_{t_0}^{(k-j-2)\alpha}, \quad 2 \leqslant k \leqslant m, \quad Y_1 = Z$$

(note that $Z^{(k)}$, $1 \leqslant k \leqslant m$, are uniquely determined, see above). Let us show that $(Y_k)_{k=1}^m$ is the solution of the Cauchy problem (3.18), (3.14). Clearly, the initial conditions are satisfied. Let us specify the derivatives Y_k'. We define

$$Y_k' := Z^{(k)} - \sum_{j=0}^{k-2} X_j \delta_{t_0}^{(k-j-1)\alpha}, \quad 1 \leqslant k \leqslant m.$$

Observe that Y_k' is indeed the derivative of Y_k. Now for every $1 \leqslant k \leqslant m-1$ we have

$$Y_k' - Y_{k+1} = X_{k-1}\delta_{t_0}^{\alpha}.$$

Further,

$$Y_m' - \sum_{k=0}^{m-1} A_k Y_{k+1} = Z^{(m)} - \sum_{j=0}^{m-2} X_j \delta_{t_0}^{(m-j-1)\alpha} - \sum_{k=0}^{m-1} A_k \left(Z^{(k)} - \sum_{j=0}^{k-1} X_j \delta_{t_0}^{(k-j-1)\alpha} \right) =$$
$$= \sum_{k=0}^{m} A_k \left(Z^{(k)} - \sum_{j=0}^{k-1} X_j \delta_{t_0}^{(k-j-1)\alpha} \right) + X_{m-1}\delta_{t_0}^{\alpha} = F + X_{m-1}\delta_{t_0}^{\alpha},$$

so $(Y_k)_{k=1}^m$ is the solution of the Cauchy problem (3.18), (3.14), as required. □

The next statement is an analogue of Theorem 3.5.

Theorem 3.5'. The solution of the Cauchy problem (3.11), (3.15) depends continuously on $F \in \mathcal{R}^{n \times n\prime}$ and $X_k \in \mathbf{C}^{n \times n}$.

Proof. The proof follows from the possibility of reduction of linear differential equation (3.11) to linear system (3.18) as in the proof of Theorem 3.4' and from the statement of Theorem 3.5 applied to system (3.18). □

Proof of Example 1.1. Let us find the solution of Cauchy problem (1.6) (the solutions of Cauchy problems (1.5) and (1.7) can be found similarly). According to Theorem 3.3, if x is the solution of Cauchy problem (1.5), then x admits the representation

$$x = e^{a\zeta_\tau} y,$$

where $e^{a\zeta_\tau(t)}$ is the fundamental solution of the corresponding homogeneous equation, and y is the primitive of the distribution $e^{-a\zeta_\tau} b\delta_\tau'^{\alpha}$ which is equal to 0 for $t < t_0$, so

$$\begin{aligned}(e^{-a\zeta_\tau} b\delta_\tau'^{\alpha}, \varphi) = (\delta_\tau'^{\alpha}, be^{-a\zeta_\tau}\varphi) = \\ &= -\alpha b\big(-ae^{-\zeta_\tau(\tau+)}\theta_\tau(\tau+)\varphi(\tau+) + e^{-\zeta_\tau(\tau+)}\varphi'(\tau+)\big) - \\ &- (1-\alpha)b\big(-ae^{-\zeta_\tau(\tau-)}\theta_\tau(\tau-)\varphi(\tau-) + e^{-\zeta_\tau(\tau-)}\varphi'(\tau-)\big) = \\ &= \alpha b(a\varphi(\tau+) - \varphi'(\tau+)) - (1-\alpha)b\varphi'(\tau-),\end{aligned}$$

so $e^{-\zeta_\tau} b\delta_\tau'^{\alpha} = ab\alpha\delta_\tau^{+} - b\delta_\tau'^{\alpha}$, and the required primitive is $ab\alpha\theta_\tau - b\delta_\tau^{\alpha}$. □

Proof of Example 1.2. First note that $a \in \mathbb{G}^\infty$, and $b \in \mathbb{L}_{\rm loc}$, so $b \in \mathcal{R}'$. As it follows from Theorem 3.3, we have to find the primitive of the distribution $e^{-\int_0^t a(a)ds} b''$ which is equal to zero for $t < 0$. Due to absolute convergence of the series $\sum_{\gamma\in\Upsilon} b_\gamma$, we have the following chain of equalities:

$$\begin{aligned}&\left(\exp\left(-\int_0^t a(a)ds\right) b'', \varphi\right) = \\ &= \left(\exp\left(-\int_0^t a(a)ds\right)\sum_{\gamma\in\Upsilon} b_\gamma\delta_\gamma'^{\alpha(\gamma)}, \varphi\right) = \sum_{\gamma\in\Upsilon} b_\gamma(\delta_\gamma'^{\alpha(\gamma)}, e^{-\int_0^t a(a)ds}\varphi) = \\ &= -\sum_{\gamma\in\Upsilon} b_\gamma e^{-\int_0^\gamma a(a)ds}\big(\alpha(\gamma)\varphi'(\gamma+) + \big(1-\alpha(\gamma)\big)\varphi'(\gamma-) - \\ &\qquad - \alpha(\gamma)a(\gamma+)\varphi(\gamma+) - \big(1-\alpha(\gamma)\big)a(\gamma-)\varphi(\gamma-)\big),\end{aligned}$$

so the required primitive is

$$\sum_{\gamma\in\Upsilon} b_\gamma \exp\left(-\int_0^\gamma a(a)ds\right)\left(\big(\alpha(\gamma)a(\gamma+) + \big(1-\alpha(\gamma)\big)a(\gamma-)\big)\theta_\gamma - \delta_\gamma^{\alpha(\gamma)}\right),$$

the rest of the proof is straightforward. □

4. Proofs of Lemmas 2.2, 2.8 and Theorem 2.5

Proof of Lemma 2.2. By definition, given an absolutely convex set $U \subset \mathcal{R}$, we have that U is a neighbourhood in $\mathcal{R}$ if and only if $U \cap \mathbb{G}^\infty(J)$ is a neighbourhood in $\mathbb{G}^\infty(J)$, for every $J \in \mathfrak{J}$. Suppose that $\varphi_k \to \varphi$ in $\mathbb{G}^\infty$ and there exists $J_0 \in \mathfrak{J}$ such that $\operatorname{supp}(\varphi_k) \subset J_0$ for all $k \in \mathbf{N}$. Now, given a neighbourhood of zero $U \subset \mathcal{R}$ and arbitrary $J \in \mathfrak{J}$, we have that

$$(\varphi_k - \varphi)|_J \in U \cap \mathbb{G}^\infty(J)$$

starting with certain k, since $U \cap \mathbb{G}^\infty(J)$ is a neighbourhood of zero in $\mathbb{G}^\infty(J)$, and $(\varphi_k - \varphi)|_J \to 0$ in $\mathbb{G}^\infty(J)$. The latter follows from the fact that every neighbourhood of zero $U_J \subset \mathbb{G}^\infty(J)$ can be extended to a neighbourhood of zero $U_I \subset \mathbb{G}^\infty$, so that $U_I \cap \mathbb{G}^\infty(J) = U_J$, while $\varphi_k - \varphi \to 0$ in $\mathbb{G}^\infty$. By definition, since $J \in \mathfrak{J}$ was arbitrary, this implies that $\varphi_k \to \varphi$ in $\mathcal{R}$.

Conversely, suppose that $\varphi_k \to \varphi$ in $\mathcal{R}$. First, observe that $\varphi_k \to \varphi$ in $\mathbb{G}^\infty$, since given any neighbourhood $U_I \subset \mathbb{G}^\infty$, $U_I \cap \mathbb{G}^\infty(J)$ is a neighbourhood in $\mathbb{G}^\infty(J)$ for every $J \in \mathfrak{J}$, so U_I is a neighbourhood in $\mathcal{R}$, and, as a result, $\varphi_k - \varphi \in U_I$ starting with certain k. Second, suppose that there is no such $J_0 \in \mathfrak{J}$, and there exists a sequence of subintervals

$$\{J_k\}_{k=1}^{\infty}, \quad J_k \subset J_{k+1}, \quad J_{k+1} \setminus J_k \neq \varnothing, \quad \cap_{k=1}^{\infty} J_k = I,$$

such that $\varphi_k-\varphi \notin \mathbb{G}^\infty(J_k)$ for all $k\in\mathbf{N}$. The latter may be possible if and only if $\operatorname{supp}(\varphi_k-\varphi)\not\subset \bar{J}_k$. Then we may specify a neighbourhood of zero $U\subset\mathcal{R}$ such that $\varphi_k-\varphi\notin U$ for every $k\in\mathbf{N}$, so $\varphi\not\to\varphi$ in $\mathcal{R}$, a contradiction. Indeed, let us denote L_k^- and L_k^+ the left-hand side and the right-hand side half-open components of $J_{k+1}\setminus J_k$. Assume without loss of generality that $L_k^-\neq\varnothing$. Let us define $t_k\in L_k^-$ to be such that $|\varphi_k(t_k-)-\varphi(t_k-)|>0$. By definition, sequence $\{t_k\}_{k=1}^\infty$ tends to the left end-point of the interval I. Further, define the monotonically decreasing sequence $\{r_k\}_{k=1}^\infty$ such that

$$0<r_k<|\varphi_k(t_k-)-\varphi(t_k-)|,\quad k\in\mathbf{N},$$

and $r_k\to 0$. Now let $r\in\mathbb{C}$ be a function such that $r(t)>0$ for all $t\in I$, and $r(t_k)=r_k$. Finally, we may define the required neighbourhood by

$$U=\{\psi\in\mathcal{R}:|\psi(t)|<r(t),t\in I\}.$$

For any $J\in\mathfrak{J}$ $\max_{t\in\bar{J}}|r(t)|>0$, so $U\cap\mathbb{G}^\infty(J)$ is a neighbourhood in $\mathbb{G}^\infty(J)$. □

In order to prove Theorem 2.5 we will need the following lemma, whose proof is identical to the proof of the analogous statement for the space $\mathcal{D}'$ in [Shi84].

Lemma 4.1. *Given $\{f_k\}_{k=1}^\infty\subset\mathcal{R}'$, $\{\varphi_k\}_{k=1}^\infty\subset\mathcal{R}$ such that $\varphi_k\to 0$ in $\mathcal{R}$ and for every $\varphi\in\mathcal{R}$ there exists $\lim(f_k,\varphi)\in\mathbf{C}$, we have that $(f_k,\varphi_k)\to 0$.*

Proof. Suppose the contrary. Then we may assume (consider a subsequence, if necessary,) that there exists $c>0$ such that $|(f_k,\varphi_k)|\geqslant c$, $k\in\mathbf{N}$. Since $\varphi_k\to 0$ in $\mathcal{R}$, we may suppose that

$$\|\varphi_k^{(j)}\|_{\mathbb{L}_\infty}\leqslant\frac{1}{4^k}$$

for all $j\leqslant k$. Let us put $\psi_k=2^k\varphi_k$. Then

$$\|\psi_k^{(j)}\|_{\mathbb{L}_\infty}\leqslant\frac{1}{2^k},\tag{4.1}$$

for all $j\leqslant k$, so $\psi_k\to 0$ in $\mathcal{R}$, though

$$|(f_k,\psi_k)|=2^k|(f_k,\varphi_k)|\geqslant 2^kc\to\infty.$$

Now let us choose f_{k_1}, ψ_{k_1} such that $|(f_{k_1},\psi_{k_1})|>1$. Suppose that f_{k_j}, ψ_{k_j} are defined, $1\leqslant j\leqslant l-1$. Suppose that for every $k\geqslant k'$ we have

$$|(f_{k_j},\psi_k)|<\frac{1}{2^{l-j}}$$

for $1\leqslant j\leqslant l-1$. Then there exists $k_l\geqslant k'$ such that

$$|(f_{k_l},\psi_{k_l})|>\sum_{j=1}^{l-1}|(f_{k_l},\zeta_{k_j})|+l\tag{4.2}$$

since $|(f_k,\psi_k)|\to\infty$, we have that $(f_k,\psi_{k_j})\to 0$ as $k\to\infty$. Suppose that the sequence $\{\psi_{k_l}\}_{l=1}^\infty$ is constructed. Let us define $\psi=\sum_{j=1}^\infty\psi_{k_j}$, where the series converges due to (4.1), so $\psi\in\mathcal{R}$. Consequently,

$$(f_{k_l},\psi)=\sum_{j=1}^{l-1}(f_{k_l},\psi_{k_j})+(f_{k_l},\psi_{k_l})+\sum_{l+1}^{\infty}(f_{k_l},\psi_{k_j}).$$

Since (4.2) and

$$\sum_{j=l+1}^{\infty}(f_{k_l},\psi_{n_j})<\sum_{j=l+1}^{\infty}\frac{1}{2^{j-l}}=1,$$

we obtain that $|(f_{k_l},\psi)|>l-1$. This contradicts to the equality $\lim(f_k,\psi)=(f,\psi)$, where $f=\lim f_k$, so the proof is complete. □

Proof of Theorem 2.5. Note that $g_k\varphi \to g\varphi$ in $\mathcal{R}$ for every $\varphi \in \mathcal{R}(\Omega)$, so

$$|(g_kf_k,\varphi)-(gf,\varphi)| = |(f_k,g_k\varphi)-(f,g\varphi)| \leqslant \\ \leqslant |(f_k,g_k\varphi)-(f_k,g\varphi)|+|(f_k,g\varphi)-(f,g\varphi)| \leqslant \\ \leqslant |(f_k,g_k\varphi-g\varphi)|+|(f_k,g\varphi)-(f,g\varphi)| \to 0$$

according to Lemma 4.1 and due to the fact that $f_k \to f$ in $\mathcal{R}'$. □

Let $\mathbb{PC} \subset \mathbb{G}$ be the space of piece-wise constant functions $I \mapsto \mathbf{C}$.

Lemma 4.2 ([Die69])**.** *$\mathbb{PC}$ is dense in $\mathbb{G}$.*

Lemma 4.3. *The subspace $\mathbb{F}$ is dense in $\mathbb{G}^\infty$. Furthermore, for every $f \in \mathbb{G}^\infty$ there exists a subsequence $\{f_l\}_{l=1}^\infty \subset \mathbb{F}$ such that $f_l \to f$ in $\mathbb{G}^\infty$, $f_l^{(l+1)} \in \mathbb{C}^\infty$, $l \in \mathbf{N}$, and*

$$\|f_l^{(j)} - f^{(j)}\|_{\mathbb{L}_\infty} < l^{-1}$$

for all $l \geqslant j$, $j \in \mathbf{N}_0$.

Proof. Let $f \in \mathbb{G}$. First, for every $k \in \mathbf{N}_0$ we define the subsequence $\{f_{lk}\}_{l=1}^\infty \subset \mathbb{F}$ such that

$$f_{lk}^{(j)} \to f^{(j)}$$

in $\mathbb{L}_\infty$ for all $0 \leqslant j \leqslant k$. Let us note that the algebra $\mathbb{PC} \subset \mathbb{F}$ is closed under the operations of integration and differentiation almost everywhere. According to Lemma 4.2 there exists a subsequence $\{p_l^k\}_{l=1}^\infty \subset \mathbb{PC}$ such that $p_l^k \to f^{(k)}$ in $\mathbb{L}_\infty$, so

$$\int_a^t p_l^k(s)ds \to \int_a^t f^{(k)}(s)ds$$

in $\mathbb{L}_\infty$. Let

$$p_l^{k-1}(t) = \int_a^t p_l^k(s)ds + q_l^{k-1}(t), \quad \{q_l^{k-1}\}_{l=1}^\infty \subset \mathbb{P}, \quad q_l^{k-1} \to f^{k-1} - \int_a^t f^{(k)}(s)ds$$

in $\mathbb{L}_\infty$, so $p_l^{k-1} \to f^{k-1}$ in $\mathbb{L}_\infty$, $(p_l^{k-1})' = p_l^k$. Further, let us define $\{p_l^{k-2}\}_{l=1}^\infty$,

$$p_l^{k-2}(t) = \int_a^t p_l^{k-1}(s)ds + q_l^{k-2}(t), \quad \{q_l^{k-2}\}_{l=1}^\infty \subset \mathbb{P}, \quad q_l^{k-2} \to f^{k-2} - \int_a^t f^{(k-1)}(s)ds$$

in $\mathbb{L}_\infty$, so $p_l^{k-2} \to f^{k-2}$ in $\mathbb{L}_\infty$, $(p_l^{k-2})' = p_l^{k-1}$. We may continue this process, and find $p_l^0 \in \mathbb{F}$.

Let us define $f_{lk} := p_l^0$. Then $f_{lk}^{(j)} = p_l^j \in \mathbb{F}$, and

$$f_{lk}^{(j)} \to f^{(j)}$$

in $\mathbb{L}_\infty$, where $0 \leqslant j \leqslant k$. We may assume that

$$\|f_{lk}^{(j)} - f^{(j)}\|_{\mathbb{L}_\infty} < \frac{1}{l}, \quad l \in \mathbf{N}, \quad 0 \leqslant j \leqslant k.$$

Now define $f_l := f_{ll} \in \mathbb{F}$ ($l \in \mathbf{N}$). Then $\|f_l^{(j)} - f^{(j)}\|_{\mathbb{L}_\infty} < \frac{1}{l}$, $l \geqslant j$, $j \in \mathbf{N}_0$, so $f_l \to f$ in $\mathbb{G}^\infty$.

The sequence $\{f_l\}_{l=1}^\infty$ constructed above is the one required, since $f_l^{(l+1)} \equiv 0$. □

Proof of Lemma 2.8. Let $\varphi \in \mathcal{R}$. According to Lemma 4.3 there exists $\{f_k\}_{k=1}^\infty \subset \mathbb{PC}$ such that $f_k \to \varphi$ in $\mathbb{G}^\infty$. Clearly, there exists a test function $\xi \in \mathcal{D}$ such that $\xi \equiv 1$ in certain open neighbourhood of $\mathrm{supp}(\varphi)$, $\xi \equiv 0$ in certain (larger) open neighbourhood of $\mathrm{supp}(\varphi)$, which possesses the compact closure in I. Then $\xi f_k \in \mathcal{R}_F$, $\xi\varphi = \varphi$, and $\xi f_k \to \xi\varphi$ in $\mathcal{R}$. Since $\varphi \in \mathcal{R}$ was chosen arbitrarily, the subspace $\mathcal{R}_F$ is dense in $\mathcal{R}$.

If $\{f_k\}_{k=1}^\infty$ is chosen as in the statement of Lemma 4.3, and we have chosen $\xi \equiv 1$ in certain neighbourhood t, then $\{\xi f_k\}_{k=1}^\infty \subset \mathcal{R}_F$ is the sequence required in the last statement. □

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